Stability under dwell time constraints: Discretization revisited

Thomas Mejstrik, and Vladimir Yu. Protasov

***Abstract*—We decide the stability and compute the Lyapunov exponent of continuous-time linear switching systems with a guaranteed dwell time. The main result asserts that the discretization method with step size h approximates the Lyapunov exponent with the precision $C\,h^2$, where C is a constant. Let us stress that without the dwell time assumption, the approximation rate is known to be linear in h. Moreover, for every system, the constant C can be explicitly evaluated. In turn, the discretized system can be treated by computing the Markovian joint spectral radius of a certain system on a graph. This gives the value of the Lyapunov exponent with a high accuracy. The method is efficient for dimensions up to, approximately, ten; for positive systems, the dimensions can be much higher, up to several hundreds.**

***Index Terms*—discretization, dynamical system on graphs, extremal norm, joint spectral radius, linear switching system, stability, Lyapunov exponent, multinorm, 49M25, 93C30, 37C20, 15A60**

I. Introduction

We consider a linear switching system of the form

$$\begin{cases} \dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t), \quad t \in [0, +\infty), \quad A(t) \in \mathcal{A} \\ \boldsymbol{x}(0) = 0 \end{cases} \tag{1}$$

with a positive dwell time restriction. This is a linear ODE on the vector-function $\boldsymbol{x} : \mathbb{R}_+ \to \mathbb{R}^d$ with a matrix function $A(t)$ taking values from a given finite set $\mathcal{A} = \{A_1, \ldots, A_n\}$ called *control set* of matrices (*regimes, modes*). The *control function*, or the *switching law* is an arbitrary piecewise constant function $A : \mathbb{R}_+ \to \mathcal{A}$ with the lengths of every stationary interval (*switching interval*) at least m, where $m > 0$ is a given dwell-time constraint. This dwell time assumption has the practical meaning that the switches between regimes are not instantaneous but take some positive time.

For the sake of simplicity we consider only the case of finite control sets $\mathcal{A}$ and of the same dwell time for all regimes A_j. All our results are easily extended to general conditions, see Remark 1.

A. Statement of the problem

Systems (1) regularly arise in engineering applications, see, for instance, [2], [16], [17], [32] and references therein. One of the main problems is to find or estimate the fastest possible growth rate of trajectories, in particular, to decide about the stability of the system. The stability problem under the dwell time restrictions have been studied in numerous work [4], [5], [7]–[9], [31], [33].

The Lyapunov exponent $\sigma = \sigma(\mathcal{A})$ of a linear switching system is the infimum of the numbers α such that for every trajectory $\boldsymbol{x}(\,\cdot\,)$, we have $\|\boldsymbol{x}(t)\| \le Ce^{\alpha t}$, $t \ge 0$, for some constant $C = C(\boldsymbol{x})$. Clearly, if $\sigma < 0$, then the system is asymptotically stable, i.e., all its trajectories tend to zero as $t \to +\infty$. The converse is also true, although less trivial [4], [21]. Thus, the asymptotic stability is equivalent to the inequality $\sigma < 0$. If, in addition, the system is irreducible, i.e., the matrices from $\mathcal{A}$ do not share common nontrivial invariant subspaces, then the inequality $\sigma \le 0$ is equivalent to the (usual) stability, when all trajectories are bounded. Let us note that if the matrices of the control set $\mathcal{A}$ have a common invariant subspace, then the computation of the Lyapunov exponent is reduced to problems in smaller dimensions by a common matrix factorization [17]. Therefore, in what follows we additionally assume that the system is irreducible.

Most of results on the stability of a linear switching systems have been obtained without the dwell time assumption. Usually it is done either by constructing a Lyapunov function [3], [11], [12], [18], [21], or by approximating of the trajectories [23], [29], [30]. The latter includes the discretization approach, which is, of course, the most obvious way to analyse ODEs. Replacing the derivative $\dot{\boldsymbol{x}}(t)$ by the divided difference $\frac{\boldsymbol{x}(t+h)-\boldsymbol{x}(t)}{h}$, we get the Euler piecewise-linear approximation. Another way of discretization is to replace the switching law by a piecewise-constant function with intervals being multiples of a given step size $h > 0$. Solving the corresponding ODE in each interval we obtain $\boldsymbol{x}(t+h) = e^{hA_j}\boldsymbol{x}(t)$, which gives a piecewise-exponential approximation of the trajectory. Both of those methods lead to a discrete time switching system of the form $\boldsymbol{y}_{k+1} = B(k)\boldsymbol{y}_k$, $k \ge 0$. For the Euler discretization, the control set $\mathcal{B}$ consists of matrices $B_j = I + hA_j$, for the second approach, $B_j = e^{hA_j}$. The Lyapunov exponent of the discrete system (we denote it by σ_h) can be efficiently computed by evaluating the *joint spectral radius* of the matrix family $\mathcal{B}$ [1]. The recent progress in the joint spectral radius problem [13], [19] allows us to calculate it with a good accuracy or, in most cases, even to find it precisely. Thus, the Lyapunov exponent $\sigma(\mathcal{A})$ of system (1) can be efficiently computed, provided it is close enough to the Lyapunov exponent σ_h of its discretization. The crucial problem is to estimate the precision, i.e., the difference $|\sigma(\mathcal{A}) - \sigma_h|$, depending on the step size h.

The discretization method in the stability problem has drawn much attention in the recent literature and several estimates for the precision $|\sigma(\mathcal{A}) - \sigma_h|$ have been obtained [12], [28]–[30].

Submitted: 22. January 2024. This work was supported by the Austrian Science Foundation (FWF) under Grant P33352-N.

Thomas Mejstrik, University of Vienna; thomas.mejstrik@gmx.at

Vladimir Yu. Protasov, University of L'Aquila, Italy; vladimir.protasov@univaq.it

For both aforementioned methods, it is linear in h, i.e., $|\sigma(\mathcal{A})-\sigma_h| \le Ch$, where, for the constant C, usually only rough upper bounds are known.

B. Overview of the main results

We establish lower and upper bounds which localize the Lyapunov exponent of the system (1) to an interval of length at most $C\,h^2$, and moreover, show that C can be explicitly found.

The main result, Theorem 5, states that the second method of discretization, when $B_j = e^{hA_j}$, leads to the double inequality $\sigma_h \le \sigma(\mathcal{A}) \le \sigma_h + C\,h^2$, where the constant C is expressed by means of the so-called *extremal multinorm*. This multinorm is constructed simultaneously with the computation of σ_h.

The estimate from Theorem 5 gives a very precise method of computation of the Lyapunov exponent $\sigma(\mathcal{A})$. The numerical results in dimensions $d \le 9$ are given in Section 6. In dimension $d = 6$ the precision is usually between 0.1 and 0.15. The computation time on our PC (5 cores, 3.6 GHz) is about one hour. For positive systems, the method performs much better even in high dimensions. For $d \le 200$, the precision does not exceed 0.01. For $d \le 100$, the computation takes a few seconds.

The use of the new estimate, however, is complicated by the fact that the stability of a discrete system under the dwell time constraint cannot be analysed by the traditional scheme. For such systems, the Lyapunov function may not exist at all [10]. The computation of the Lyapunov exponent requires the concept of *restricted* or *Markovian* joint spectral radius [6], [15], [24], [26], which are special cases of the recent theory of dynamical system on graphs developed in [10], [22], [25]. That is why we need to do preliminary work to introduce the system on graphs and the concept of Lyapunov multinorm.

Remark 1. Our estimates for the approximation rate do not include the number of matrices, which makes them applicable for arbitrary compact control set $\mathcal{A}$. They are also easily generalized to mode-dependent constraints on the dwell time, when m is a function of the matrix $A \in \mathcal{A}$. Finally, the results can be extended to mixed (discrete-continuous) systems with hybrid control. In particular, when every switching from the regime A_i to A_j is realized by a given linear operator E_{ji} that can be different from $e^{\,mA_j}$.

C. The structure of the paper

In Section II we introduce auxiliary facts and notation such as Markovian joint spectral radius, dynamical systems, and Lyapunov multinorms. The fundamental theorem and corollaries are formulated and discussed in Sections III, the proofs are given in Section IV. Sections V and VI present the algorithm and analyse numerical properties.

Throughout the paper we denote vectors by bold letters, I is the identity matrix, $\rho(X)$ is the spectral radius of the matrix X, which is the maximum modulus of its eigenvalues.

II. Preliminary facts. Dynamical systems on graphs

The discretization with step size h approximates the system (1) with the dwell time parameter $m > 0$ by the discrete-time system $\boldsymbol{y}_{k+1} = B(k)\boldsymbol{y}_k$, where for each $k \ge 0$, the matrix $B(k)$ is either e^{hA_j} or e^{mA_j}, $A_j \in \mathcal{A}$, see Definition 4 below. The dwell time assumption imposes a firm restriction to the switching law: It must be a sequence of blocks of the form $B(k+N)\cdots B(k+1)B(k) = (e^{hA_j})^N e^{mA_j}$, where $j \in \{1,\dots,n\}$, the length $N \ge 0$ depends on the block, and two neighbouring blocks must have different modes j. Thus, each block has to begin with e^{mA_j} followed by a power of e^{hA_j}, and then switch to the next block with a different j. The general theory of discrete systems with constraints on switching laws has been developed in [6], [10], [15], [26] and then extended to general dynamical systems on graphs. We begin with necessary definitions and notation.

Consider a directed graph G with vertices $v_1,\dots,v_n$ and edges ℓ_{ji}, which may include loops ℓ_{ii}. Each vertex v_i is associated to a finite-dimensional linear space V_i. To every edge ℓ_{ji}, if it exists, we associate a linear operator $E_{ji} : V_i \to V_j$ and a positive number h_{ji}, which is referred to as the *time of action* of E_{ji}.

Definition 2. *The* d*ynamical system on the graph G is the equation $\boldsymbol{x}_{k+1} = E(k)\boldsymbol{x}_k$, on the sequence $\{\boldsymbol{x}_k\}_{k\ge 0}$, where for each k, the operator $E(k)$ is chosen from the set $\{E_{ji}\}_{j=1}^n$ if $\boldsymbol{x}_k \in V_i$. The point $\boldsymbol{x}_k$ corresponds to the time t_k and $t_{k+1} = t_k + h_{ji}$.*

The usual discrete-time linear switching system $\boldsymbol{x}_{k+1} = B(k)\boldsymbol{x}_k$, $B(k) \in \mathcal{B} = \{B_j\}_{j=1}^n$, corresponds to the case when G is a complete graph, V_j are all equal to $\mathbb{R}^d$, every vertex v_j has n incoming edges $\ell_{ji}, i = 1,\dots,n$, with the operators E_{ji} equal to the same operator B_j, and all time intervals h_{ji} are equal to 1.

Let us have an arbitrary dynamical system on a graph. We assume that each space V_j is equipped with a certain norm $\|\cdot\|_j$. The collection of those norms $\{\|\cdot\|_j\}_{j=1}^n$ is called a *multinorm*. We use the short notation $\|\boldsymbol{x}\|$ meaning that $\|\boldsymbol{x}\| = \|\boldsymbol{x}\|_j$ for $\boldsymbol{x} \in V_j$.

Every trajectory $\{\boldsymbol{x}_k\}_{k=0}^\infty$ of the system on G corresponds to an infinite path $v_{i_0} \to v_{i_1} \to \cdots$, where $\boldsymbol{x}_0 \in V_{i_0}$ is a starting point and $\boldsymbol{x}_{k+1} = E_{i_{k+1}i_k}\boldsymbol{x}_k$, $k \ge 0$. Thus, $\boldsymbol{x}_k \in V_{i_k}$ for every k. Every point $\boldsymbol{x}_k$ corresponds to the time $t_k = \sum_{s=1}^k h_{i_s i_{s-1}}$, which is the total time of the way from v_{i_0} to v_{i_k}.

The *(Markovian) joint spectral radius* $\hat\rho = \hat\rho(\mathcal{A})$ of the system is

$$\hat\rho = \lim_{k\to\infty} \max_{\{\boldsymbol{x}_s\}_{s=0}^k} \|\boldsymbol{x}_k\|^{1/t_k},$$

where the maximum is computed over all trajectories $\{\boldsymbol{x}_s\}_{s\ge 0}$ with $\|\boldsymbol{x}_0\| = 1$. Thus, for every trajectory, we have $\|\boldsymbol{x}_k\| \le C\hat\rho^{t_k}$, $k \in \mathbb{N}$. The joint spectral radius is the rate of the fastest growth of trajectories. See [10] for the correctness of the definition and for basic properties of this notion.

Let us now consider an arbitrary cycle of the graph G: $v_{i_0} \to v_{i_1} \to \cdots \to v_{i_k} = v_{i_0}$. Denote by Π the product of operators $E_{i_k i_{k-1}} \cdots E_{i_1 i_0}$ along the cycle. We have $\Pi\boldsymbol{x}_0 = \boldsymbol{x}_k$.

For the spectral radius $\rho(\Pi)$, which is the maximal modulus of eigenvalues of Π, we have $[\rho(\Pi)]^{1/t_k} \le \hat{\rho}$. Indeed, the left-hand side is the rate of growth of a periodic trajectory going along that cycle, and it does not exceed the maximal rate of growth $\hat{\rho}$ over all trajectories. On the other hand, there are cycles for which the left-hand side is arbitrarily close to $\hat{\rho}$ [10], [15].

Definition 3. *A multinorm is called* extremal *for a system on a graph if for each $i, j \in \{1, \dots, n\}$ and for every $\boldsymbol{x} \in V_i$, we have $\|E_{ji}\boldsymbol{x}\|_j \le \hat{\rho}^{h_{ji}}\|\boldsymbol{x}\|_i$, provided the edge ℓ_{ji} exists.*

For the extremal multinorm, for every trajectory, we have $\|\boldsymbol{x}_k\| \le \hat{\rho}^{t_k}\|\boldsymbol{x}_0\|$, $k \in \mathbb{N}$.

The extremal multinorm always exists, provided the system is irreducible. The reducibility means the existence of subspaces $V_j' \subset V_j$, $i = 1, \dots, n$, where at least one subspace is nontrivial and at least one inclusion is strict, such that $E_{ji}V_i' \subset V_j'$, whenever the edge ℓ_{ji} exists.

Theorem A [10] *An irreducible dynamical system on a graph possesses an extremal multinorm.*

Theorem A implies, in particular, that for every irreducible system, we have $\hat{\rho} > 0$. Otherwise, $E_{ji}\boldsymbol{x} = 0$ for all $\boldsymbol{x} \in V_i$, i.e., all E_{ji} are equal to zero, in which case the system is clearly reducible. Therefore, one can always normalize the operators as $\tilde{E}_{ji} = \hat{\rho}^{-h_{ji}}E_{ji}$, after which the new dynamical system (on the same graph and with the same time intervals) has the joint spectral radius equal to one. Moreover, it possesses the same extremal norm, for which $\|\tilde{E}_{ji}\boldsymbol{x}\|_j \le \|\boldsymbol{x}\|_i$, hence, for every trajectory $\tilde{\boldsymbol{x}}_k$, the sequence $\|\tilde{\boldsymbol{x}}_k\|$ is non-increasing in k.

Now we are going to define the discretization of the dwell time constrained system (1) and present it as a system on a suitable graph. Then we apply Theorem A and use an extremal multinorm to estimate the approximation rate.

Definition 4. *The* h-discretization of the system $\mathcal{A}$ with step size h *is a system $\mathcal{A}_h$ on a complete graph with n vertices, in which $V_j = \mathbb{R}^d$, $E_{jj} = e^{hA_j}$, $h_{jj} = h$ for all $j = 1, \dots, n$, and $E_{ji} = e^{mA_j}$, $h_{ji} = m$ for all pairs (i, j), $i \ne j$.*

Thus, the h-discretization $\mathcal{A}_h$ is a dynamical system on a complete graph, where every vertex v_j has $n-1$ incoming edges from all other vertices, all associated to the operator e^{mA_j} with the time of action m, and also has a loop associated to e^{hA_j} with time of action h. Respectively, a multinorm $\|\cdot\| = \{\|\cdot\|_j\}_{j=1}^n$ can now be interpreted as a collection of norms in $\mathbb{R}^d$, each associated to the corresponding regime A_j. See Figure 1 for an example of a system $\mathcal{A}$ with three matrices.

The h-discretization can also be presented as a discrete-time linear switching system in $\mathbb{R}^d$: $\boldsymbol{x}_{k+1} = B(k)\boldsymbol{x}_k, k \ge 0$, where the sequence $B(k)$ has values from the control set $\{e^{mA_j}, e^{hA_j}\}_{j=1}^n$ with the following restriction:

Every element $B(k)$ equal to e^{mA_j} or e^{hA_j} can be followed by either e^{hA_j} or e^{mA_s}, $s \ne j$.

The operators e^{mA_j}, e^{hA_j} act during the time intervals of lengths m and h respectively. We denote by $\hat{\rho}(\mathcal{A}_h)$ the (Markovian) joint spectral radius of the system $\mathcal{A}_h$. By Theorem A, if the family $\mathcal{A}$ is irreducible, then for every h its h-discretization possesses an extremal norm. Note that this norm can be different for different h.

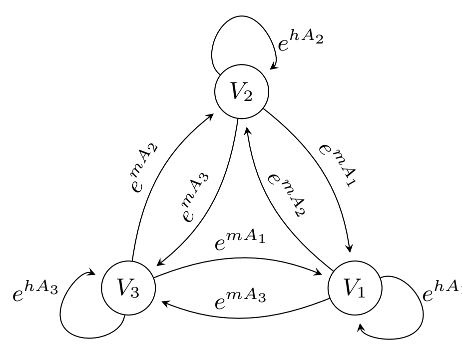

Fig. 1. The dynamical system on graph, $n = 3$.

III. The fundamental theorem

Now we are formulating the main result. We consider a linear switching system $\mathcal{A}$ given by (1) and its h-discretization $\mathcal{A}_h$. The joint spectral radius of $\mathcal{A}_h$ is denoted by $\hat{\rho}(\mathcal{A}_h)$.

Theorem 5. *Let $\mathcal{A}$ be an irreducible continuous-time linear switching system with the dwell time constraint m. Then for every discretization step $h > 0$, we have*

$$\sigma_h \le \sigma(\mathcal{A}) \le \sigma_h - \frac{1}{m}\ln\left(1 - \frac{\|(\mathcal{A} - \sigma_h I)^2\|}{8}h^2\right), \quad (2)$$

where $\sigma_h = \ln\hat{\rho}(\mathcal{A}_h)$, $\|\cdot\| = \{\|\cdot\|_j\}_{j=1}^n$ is an extremal multinorm of $\mathcal{A}_h$, and

$$\|(\mathcal{A} - \sigma_h I)^2\| = \max_{j=1,\dots,n}\|(A_j - \sigma_h I)^2\|_j.$$

Remark 6. The estimate (2) uses two values: the joint spectral radius of the h-discretization $\hat{\rho}(\mathcal{A}_h)$ and the operator norms of $(A_j - \sigma_h I)^2$ in the jth component of the extremal multinorm $\{\|\cdot\|_j\}$ for $\mathcal{A}_h$. They are both found by the invariant polytope algorithm [10]. We give a brief description in Section 5.

Writing the Taylor expansion of the logarithm up to the second order, we obtain the following

Corollary 7. *Under the assumptions of Theorem 5, we have*

$$\sigma_h \le \sigma(\mathcal{A}) \le \sigma_h + \frac{\|(\mathcal{A} - \sigma_h I)^2\|}{8m}h^2 + \mathcal{O}(h^4), \quad \text{as } h \to 0. \quad (3)$$

Remark 8. If $h = m$, then we have basically the discretization of an unrestricted system, and (3) becomes

$$\sigma_h \le \sigma(\mathcal{A}) \le \sigma_h + \frac{\|(\mathcal{A} - \sigma_h I)^2\|}{8}h + \mathcal{O}(h^3), \quad \text{as } h \to 0,$$

which again reveals the linear dependence on the discretization step h for unrestricted systems.

Remark 9. If $||\cdot||$ is an approximation of an extremal multinorm of $\mathcal{A}_h$ up to a factor of $1 + \varepsilon$ i.e. (in the notation of Definition 3)

$$\hat{\rho}^{h_{ji}}\|\boldsymbol{x}\|_i \le \|E_{ji}\boldsymbol{x}\|_j \le (1 + \varepsilon)\hat{\rho}^{h_{ji}}\|\boldsymbol{x}\|_i,$$

then (3) becomes

$$\sigma_h^- \le \sigma(\mathcal{A}) \le \sigma_h^+ - \frac{1}{m}\ln\left(1 - \frac{\|(\mathcal{A}-\sigma_h^- I)^2\|}{8}h^2\right),$$

where $\sigma_h^- = \ln\hat\rho(\mathcal{A}_h)$, and $\sigma_h^+ = \ln(1+\varepsilon)\hat\rho(\mathcal{A}_h)$.

The quadratic rate of approximation in formula (3) is quite unexpected since the trajectories of the continuous-time system are approximated by the trajectories of its h-discretization only with the linear rate. Nevertheless, the approximation of the Lyapunov exponent is quadratic.

Remark 10. The performance of the estimate (2) can be spoiled in two cases: either the dwell time m is too small, or the operator norm of $(A_j - \sigma_h I)^2$ is too large. Note also that (2) is an a posteriori estimate since the operator norm depends on h and is not known in advance.

Theorem 5 yields the following stability conditions for system (1):

Corollary 11. *If an h-discretization $\mathcal{A}_h$ is unstable, then so is the system $\mathcal{A}$. If $\mathcal{A}_h$ is stable and $\hat\rho \le \bigl(1 - \frac{h^2\|(\mathcal{A}-(\ln\rho)I)^2\|}{8}\bigr)^{1/m}$, then $\mathcal{A}$ is stable.*

Proof. The first inequality in (2) implies that if $\hat\rho > 1$ and hence $\sigma_h > 0$, then $\sigma(\mathcal{A}) > 0$. The converse is established similarly. □

IV. Proofs of the main results

The proof of Theorem 5 is based on a simple geometrical argument. If a curve connects two ends of a segment of length h, then the distance from this curve to the segment does not exceed h^2 multiplied by the curvature and by a certain constant. The nontrivial moment is that we need this property in an arbitrary norm in $\mathbb{R}^d$ and need to evaluate the constant depending on this norm. Then we apply this fact to each component $\|\cdot\|_j$ of the extremal norm of the system $\mathcal{A}_h$ and estimate the growth of trajectory of the system $\mathcal{A}$.

Lemma 12. *Let $\|\cdot\|$ be an arbitrary norm in $\mathbb{R}^d$ and $\boldsymbol{x} : [0,h] \to \mathbb{R}^d$ be a C^2-curve. Then, for every $\tau \in [0,h]$, the distance from the point $\boldsymbol{x}(\tau)$ to the segment $[\boldsymbol{x}(0), \boldsymbol{x}(h)]$ does not exceed $\frac{h^2}{8}\|\ddot{\boldsymbol{x}}\|_{C[0,h]}$.*

Proof. Denote by $\|\cdot\|^*$ the dual norm in $\mathbb{R}^d$, thus $\|\boldsymbol{y}\|^* = \sup_{\|\boldsymbol{x}\|=1}(\boldsymbol{y},\boldsymbol{x})$. Let $\boldsymbol{x}(0) = \boldsymbol{0}$ and $\boldsymbol{x}(h) = \boldsymbol{a}$. Let $\boldsymbol{x}(\xi)$ be the most distant point of the arc $\{\boldsymbol{x}(\tau), \tau \in [0,h]\}$ to the segment $[\boldsymbol{0},\boldsymbol{a}]$ and let this maximal distance be equal to r. Suppose $\boldsymbol{y}$ is the closest to $\boldsymbol{x}(\xi)$ point of that segment and denote $\boldsymbol{s} = \boldsymbol{x}(\xi) - \boldsymbol{y}$. Thus, $\|\boldsymbol{s}\| = r$. The segment $[\boldsymbol{0},\boldsymbol{a}]$ does not intersect the interior of the ball of radius r centred at $\boldsymbol{x}(\xi)$. Therefore, by the convex separation theorem, there exists a linear functional $\boldsymbol{p} \in \mathbb{R}^d$, $\|\boldsymbol{p}\|^* = 1$, which is non-positive on that segment, non-negative on the ball, and such that $(\boldsymbol{p},\boldsymbol{s}) = \|\boldsymbol{s}\|$. Since the point $\boldsymbol{y}$ belongs to the ball, it follows that $(\boldsymbol{p},\boldsymbol{y}) = 0$. We have $(\boldsymbol{p},\boldsymbol{s}) = (\boldsymbol{p},\boldsymbol{x}(\xi)) - (\boldsymbol{p},\boldsymbol{y}) = (\boldsymbol{p},\boldsymbol{x}(\xi))$, therefore, $(\boldsymbol{p},\boldsymbol{x}(\xi)) = r$, see Figure 2.

Defining the function $f(t) = \bigl(\boldsymbol{p},\boldsymbol{x}(t)\bigr)$ we obtain

$$f(\xi) - f(0) = \bigl(\boldsymbol{p}, \boldsymbol{x}(\xi) - \boldsymbol{x}(0)\bigr) = \bigl(\boldsymbol{p},\boldsymbol{x}(\xi)\bigr) = r. \qquad (4)$$

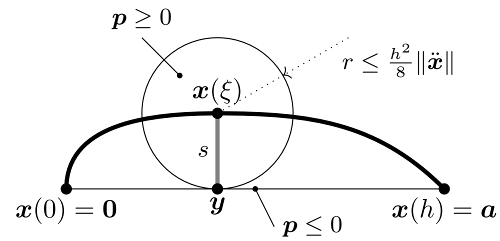

Fig. 2. Construction in proof of Lemma 12.

Without loss of generality we assume that $\xi \le \frac{1}{2}h$, otherwise one can interchange the ends of the segment $[0,h]$. The Taylor expansion of f at the point ξ gives $f(t) = f(\xi) + f'(\xi)(t-\xi) + \frac{1}{2}f'(\eta)(t-\xi)^2$, where $\eta \in [t,\xi]$. The maximum of $f(t)$ is attained at $t = \xi$, hence, $f'(\xi) = 0$. For $t = 0$, this yields $f(0) = f(\xi) + \frac{1}{2}f''(\eta)\xi^2$. Combining with (4), we obtain

$$r = -\frac{1}{2}f''(\eta)\xi^2 = \frac{1}{2}\bigl(\boldsymbol{p}, -\ddot{\boldsymbol{x}}(\eta)\bigr)\xi^2 \le \frac{1}{2}\|\ddot{\boldsymbol{x}}\|\xi^2 \le \frac{h^2}{8}\|\ddot{\boldsymbol{x}}\|.$$

which completes the proof. □

Theorem 13. *Let $\boldsymbol{x}(t)$ be a solution of the differential equation $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ with a constant $d \times d$ matrix A, $\|\cdot\|$ be an arbitrary norm in $\mathbb{R}^d$, and $h \in \bigl(0, \sqrt{8/\|A^2\|}\bigr)$ be a number. Then for every $\tau \in [0,h]$, we have*

$$\bigl\|\boldsymbol{x}(\tau)\bigr\| \le \frac{1}{1 - \frac{h^2}{8}\|A^2\|}\max\Bigl\{\|\boldsymbol{x}(0)\|, \|\boldsymbol{x}(h)\|\Bigr\}. \qquad (5)$$

Proof. It suffices to consider the vector $\boldsymbol{x}(\tau)$ with the maximal norm over all $\tau \in [0,h]$. Since $\ddot{\boldsymbol{x}} = A\dot{\boldsymbol{x}} = A^2\boldsymbol{x}$, it follows that $\|\ddot{\boldsymbol{x}}\|_{C[0,h]} = \max_{t\in[0,h]}\|A^2\boldsymbol{x}(t)\| \le \|A^2\| \cdot \|\boldsymbol{x}(\tau)\|$. Hence, by Lemma 12, the distance from the point $\boldsymbol{x}(\tau)$ to the closest point $\boldsymbol{y}$ of the segment $[\boldsymbol{x}(0),\boldsymbol{x}(h)]$ does not exceed $\frac{h^2}{8}\|A^2\| \cdot \|\boldsymbol{x}(\tau)\|$. On the other hand, this distance is not less than $\|\boldsymbol{x}(\tau)\| - \|\boldsymbol{y}\|$. It remains to note that $\|\boldsymbol{y}\| \le \max\{\|\boldsymbol{x}(0)\|, \|\boldsymbol{x}(h)\|\}$, which follows from the convexity of the norm. Thus,

$$\frac{h^2}{8}\|A^2\| \cdot \|\boldsymbol{x}(\tau)\| \ge \|\boldsymbol{x}(\tau)\| - \max\Bigl\{\|\boldsymbol{x}(0)\|, \|\boldsymbol{x}(h)\|\Bigr\}.$$

Expressing $\|\boldsymbol{x}(\tau)\|$ we arrive at (5). □

Proof of Theorem 5. The lower bound follows trivially. To prove the upper bound, we first assume that $\hat\rho(\mathcal{A}_h) = 1$. By Theorem A, the system $\mathcal{A}_h$ possesses an extremal multinorm $\|\cdot\| = \{\|\cdot\|_j\}_{j=1}^n$. For each $j \le n$, we denote $\alpha_j = -\frac{1}{m}\ln\Bigl(1 - \frac{\|A_j^2\|_j}{8}h^2\Bigr)$. Let us show that for every $i \ne j$ and for every point $\boldsymbol{z}_0 \in V_i$, the trajectory $\boldsymbol{z}(t)$ generated in V_j by the ODE $\dot{\boldsymbol{z}} = A_j\boldsymbol{z}$ and starting at $\boldsymbol{z}_0$ possesses the property: $\|\boldsymbol{z}(t)\|_j \le e^{\alpha_j t}\|\boldsymbol{z}_0\|_j$ for all $t \ge m$. Since the multinorm is extremal, it follows that for every integer $k \ge 0$, one has $\bigl\|\boldsymbol{z}(m+kh)\bigr\|_j = \bigl\|e^{mA_j}\bigl(e^{hA_j}\bigr)^k\boldsymbol{z}_0\bigr\|_j \le \|\boldsymbol{z}_0\|_j$. Let k be the maximal integer such that $t \ge m + kh$. Thus, $t = m + kh + \tau$, $\tau \in [0,h)$. Consider the arc of the trajectory $\boldsymbol{z}(\,\cdot\,)$ on the time interval $[m+kh, m+(k+1)h]$ and denote $\boldsymbol{x}(0) = \boldsymbol{z}(m+kh)$, $\boldsymbol{x}(h) = \boldsymbol{z}(m+kh+h)$. Observe that both $\|\boldsymbol{x}(0)\|$ and $\|\boldsymbol{x}(h)\|$ do not exceed $\|\boldsymbol{z}_0\|$. Applying Theorem 13 we obtain

$$\|\boldsymbol{z}(t)\|_j = \|\boldsymbol{x}(\tau)\|_j \leq \frac{1}{1 - \frac{h^2}{8}\|A_j^2\|_j}\|\boldsymbol{z}_0\|_j$$
$$= e^{\alpha_j m}\|\boldsymbol{z}_0\|_j \leq e^{\alpha_j t}\|\boldsymbol{z}_0\|_j.$$

Using the multinorm notation and setting $\alpha = \max_{j=1}^n \alpha_j$, we conclude that for every trajectory $\boldsymbol{z}(\,\cdot\,)$ generated by one regime it holds that $\|\boldsymbol{z}(t)\| \leq e^{\alpha t}\|\boldsymbol{z}(0)\|$ for all $t \geq m$.

Consider now an arbitrary trajectory $\boldsymbol{x}(\,\cdot\,)$ of the system $\mathcal{A}$. If it does not have switches, then it is generated by some regime A_j, and the proof follows immediately from the inequality above. Let it have the switching points $t_0 < t_1 < \ldots$. This set can be infinite or finite. Applying the inequality above to arbitrary $t \in [t_k, t_{k+1}]$ and denoting $\boldsymbol{z}(0) = \boldsymbol{x}(t_k)$, we obtain $\|\boldsymbol{x}(t)\| \leq e^{\alpha(t-t_k)}\|\boldsymbol{x}(t_k)\|$. Now take a time interval $[0, T]$ and let t_N be the largest switching point on it. Applying our inequality successively for all switching intervals, we get

$$\|\boldsymbol{x}(T)\| \leq e^{\alpha(t-t_N)} \prod_{k=0}^{N-1} e^{\alpha(t_{k+1}-t_k)}\|\boldsymbol{x}(t_0)\|$$
$$\leq e^{\alpha(T-t_0)}\|\boldsymbol{x}(t_0)\|.$$

Thus, for every trajectory $\boldsymbol{x}(\,\cdot\,)$, we have $\|\boldsymbol{x}(T)\| = \mathcal{O}(e^{\alpha t})$ as $T \to \infty$. Let us remember that the norm $\|\cdot\|$ can be different on different switching intervals. Nevertheless, they all belong to the finite set of norms $\{\|\cdot\|_j\}_{j=1}^n$ which are all equivalent. Therefore, $\sigma(\mathcal{A}) \leq \alpha$.

This concludes the proof for the case $\hat{\rho} = 1$. The general case follows from this one by normalization: We replace the family $\mathcal{A}$ by $\tilde{\mathcal{A}} = \mathcal{A} - \sigma_h I$. Then $\hat{\rho}(e^{h\tilde{\mathcal{A}}}) = 1$ and $\sigma(\tilde{\mathcal{A}}) = \sigma(\mathcal{A}) - \sigma_h$. Finally, applying the theorem for the system $\tilde{\mathcal{A}}$ and substituting to (2), we complete the proof. ☐

Now, to put Theorem 5 into practise, we need to compute the value $\hat{\rho}(\mathcal{A}_h)$ and construct an extremal multinorm. We will do it in Section 5 for a general system on a graph.

V. Computing the joint spectral radius and an extremal multinorm for a system on a graph

Theorem 5 gives a recipe to compute the Lyapunov exponent of a linear switching system (1) with sufficiently high accuracy. Due to the quadratic dependence of h in (3) one can make the distance between the upper and lower bounds small by choosing an appropriate discretization step h. This plan requires solving two problems: (i) Compute the value of $\hat{\rho}(\mathcal{A}_h)$. (ii) Construct an extremal multinorm for $\mathcal{A}_h$. Both are solved simultaneously by the invariant polytope algorithm *(ipa)* derived in [13]. The ipa computes the joint spectral radius of several matrices by constructing an extremal norm. In [10] the invariant polytope algorithm was extended to discrete time systems with restrictions and to systems of graphs. We present the main idea of the algorithm in this section; for details see [10], [13], [14], [19]. The reference implementation of the algorithm can be found at *gitlab.com/tommsch/ttoolboxes*.

The (Markovian) invariant polytope algorithm (*ipa*) finds the joint spectral radius $\hat{\rho}(\mathcal{A}_h)$ and an extremal multinorm. It consists of two steps.

a) Step 1.: We fix a number N (not very large) and exhaust all cycles $v_{i_0} \to v_{i_1} \to \cdots \to v_{i_L} = v_{i_0}$ of G of length at most N. To each cycle we denote by $\Pi = E_{i_L i_{L-1}} \cdots E_{i_1 i_0}$ the product of the linear operators associated to its edges and by $T = \sum_{k=1}^{L} h_{i_k i_{k-1}}$ its total time. Recall that $E_{jj} = e^{hA_j}$, $h_{jj} = h$ for all j and $E_{ji} = e^{mA_j}$, $h_{ji} = m$ for all pairs (i, j), $i \neq j$.

We choose a cycle with the biggest value $r = \rho(\Pi)^{1/T}$ and call it *leading cycle* and respectively *leading product* Π. We set $\tilde{A}_j = A_j - (\ln r)I$, $j = 1, \ldots, n$. For the new system $\tilde{\mathcal{A}}_h$, we have $\rho(\tilde{\Pi}) = \rho(\tilde{E}_{i_L i_{L-1}} \cdots \tilde{E}_{i_1 i_0}) = 1$.

In many cases to find the leading cycle we can avoid the exhaustion and use the auxiliary Algorithm 18. It is an adaptation of the modified Gripenberg algorithm from [19] to our setting. Some of its numerical properties are assessed in Appendix A.

For the sake of simplicity of exposition, we assume that the leading eigenvector of $\tilde{\Pi}$ is positive and thus equal to one. The general case is considered similarly, see [10], [20].

Denoting by $\boldsymbol{x}_0$ the leading eigenvector of $\tilde{\Pi}$, it follows that $\boldsymbol{x}_0 \to \boldsymbol{x}_1 \to \cdots \to \boldsymbol{x}_{L-1} \to \boldsymbol{x}_0$ is the periodic trajectory corresponding to that cycle.

b) Step 2.: We try to prove that actually $\hat{\rho}(\mathcal{A}_h) = r$ and, if so, to find an extremal norm.

For every j, we denote by $\mathcal{V}_j^{(0)}$ the set of points $\boldsymbol{x}(s)$, $s = 0, \ldots, L-1$ that belong to V_j. If there are no such points, then $\mathcal{V}_j^{(0)} = \emptyset$. Suppose after k iterations we have finite sets $\mathcal{V}_j^{(k)} \subset V_j$, $j = 1, \ldots, n$. Denote by $\mathrm{co}_s\, \mathcal{V}_j^{(j)}$ the convex hull of the set $\mathcal{V}_j^{(k)} \cup (-\mathcal{V}_j^{(k)})$. Now for every j, we add to $\mathcal{V}_j^{(k)}$ all points of the sets $e^{m\tilde{A}_j}\mathcal{V}_s^{(k)}$ for all $s \neq j$ and of the set $e^{h\tilde{A}_j}\mathcal{V}_j^{(k)}$. We add only those points that do not belong to $\mathrm{co}_s\, \mathcal{V}_j^{(j)}$, the others are redundant and we discard them. This way we obtain the sets $\mathcal{V}_j^{(k+1)}$, $j = 1, \ldots, n$. We do this until $\mathcal{V}_j^{(k+1)} = \mathcal{V}_j^{(k)}$ for all j, in which case the algorithm terminates. We conclude that $\hat{\rho}(\mathcal{A}_h) = \rho(\Pi)^{1/T}$ and the Minkowski norms of the polytopes $\{\mathrm{co}_s\, \mathcal{V}_j^{(j)}\}_{j=1}^n$ form an extremal multinorm for $\mathcal{A}_h$.

Remark 14. To find the Lyapunov exponent $\sigma(\mathcal{A})$, we need to compute the operator norms $\|A_j - \sigma_h I\|_j$. Since the unit balls of the norms $||\cdot||_j$ are polytopes, this can be done efficiently by solving an LP problem [13].

To achieve a good precision one needs to choose an appropriate step size h which, however, cannot be too small. Otherwise, the matrices e^{hA_j} will be close to the identity complicating the computation of the joint spectral radius [12]. In particular, the length of the leading cycle may get too large to be handled efficiently or cannot be found at all, see Example 17. In most cases h cannot be chosen less than 0.1 (as from our numerical tests).

A. Positive systems

A continuous-time linear switching system is called *positive* if every trajectory $\boldsymbol{x}(t)$ starting in the positive orthant $\mathbb{R}_+^d$ remains in $\mathbb{R}_+^d$ for all t. Positive systems have been studied widely in literature due to many applications.

TABLE I
COMPUTATION OF THE LYAPUNOV EXPONENT FOR ARBITRARY SYSTEMS AND FOR POSITIVE SYSTEMS

Random matrices			Random Metzler matrices		
dim	ub − lb	time	dim	ub − lb	time
2	0.016512	$4s$	2	0.018851	$1s$
3	0.020846	$58s$	5	0.008385	$1s$
4	0.036887	$780s$	13	0.007760	$2s$
5	0.108899	$2300s$	34	0.007225	$4s$
6	0.133852	$4000s$	89	0.009330	$14s$
7	0.234899	$3300s$	144	0.006306	$41s$
8	0.314734	$4900s$	233	0.005352	$150s$
9	0.409434	$3500s$	377	0.005544	$560s$

The positivity of the system is equivalent to that all the matrices A_j are Metzler, i.e. all off-diagonal entries are nonnegative. In this case all the matrices e^{hA_j}, e^{mA_j} of the discrete-time system $\mathcal{A}_h$ are nonnegative. Hence, the matrix Π in the invariant polytope algorithm is also nonnegative, and therefore, the Perron-Frobenius theorem implies the nonnegativity of the leading eigenvector $\boldsymbol{x}_0$. Consequently, all the sets $\mathcal{V}_j^{(k)}$ are nonnegative, i.e., the algorithm runs entirely in $\mathbb{R}^d_+$. It is then possible to replace the polytopes $\text{co}_s\, \mathcal{V}_j^{(j)}$ by the positive polytopes $\text{co}_+\, \mathcal{V}_j^{(j)} = \{\boldsymbol{x} \geq 0 : \boldsymbol{x} \leq \text{co}_s\, \mathcal{V}_j^{(j)}\}$, where the inequalities are understood element wise.

Since the positive polytopes $Q_j^{(k)}$ are in general much larger than $P_j^{(k)}$, they absorb more points in each iteration. This reduces significantly the number of vertices of the polytopes and, respectively, the complexity of each iteration. In fact, this modification of the invariant polytope algorithm for positive systems works very efficiently even for large dimensions d.

VI. NUMERICAL RESULTS

We demonstrate the performance of estimate (2) from Theorem 5 to the computation of the Lyapunov exponent. Given pairs of random matrices and random dwell time $m \in (0, 1)$, Table I presents the performance of the Markovian ipa. For each dimension d, we conducted about 15 tests and we give the median of the best achieved accuracies i.e. the maximal lower bound from Equation (3). As test matrices we used (lhs) matrices with normally distributed entries, and (rhs) Metzler matrices with random integer entries in $[-9, 9]$. To make the examples more interesting, we omit simple cases when one matrix dominates others (which often occurs). To this end, the matrices got normalized such that the 2-norm is equal to 1.

One can see that the Markovian ipa can compute bounds in reasonable time (although the needed time depends very strongly on the matrices) up to dimension ≈ 10 for general matrices, and up to dimension ≈ 500 for Metzler matrices.

Remark 15. The scripts used to obtain the experimental results can be found at *gitlab.com/tommsch/ttoolboxes/-/tree/master/demo/dwelltime*.

A. Examples

We begin with a simple two-dimensional example illustrating the Lyapunov exponent computation by the estimates of Theorem 5 and the Markovian ipa (Section 5).

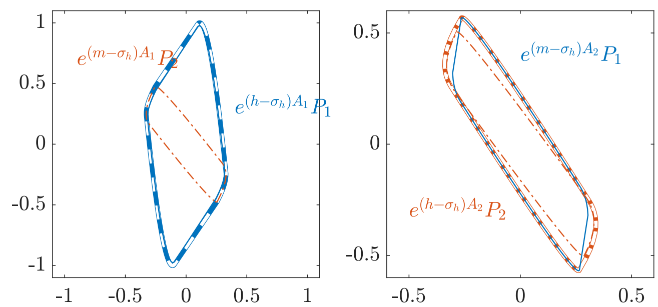

Fig. 3. Polytopes from Example 16.

Example 16. *Given dwell time $m = 1$, two matrices*

$$A_1 = \frac{1}{\sqrt{2}+2}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \frac{1}{\sqrt{2}+2}\begin{bmatrix} -2 & -2 \\ -1 & -2 \end{bmatrix},$$

and discretization step $h = 0.2$. The product $\Pi = (e^{mA_2})^1 (e^{hA_2})^7 (e^{mA_1})^1 (e^{hA_1})^{180} (e^{mA_2})^1 (e^{hA_2})^7 (e^{mA_1})^1 (e^{hA_1})^{181} (e^{mA_2})^1 (e^{hA_2})^7 (e^{mA_1})^1 (e^{hA_1})^{183}$ is a leading cycle. The ipa gives $\hat{\rho}(\mathcal{A}_h) = 1.0331...$, or respectively $\sigma_h = 0.0325...$, and polytopes P_1, $P_2 \subseteq \mathbb{R}^2$ such that $e^{(m-\sigma_h)A_1}P_2 \subseteq P_1$, $e^{(h-\sigma_h)A_1}P_1 \subseteq P_1$, $e^{(h-\sigma_h)A_2}P_2 \subseteq P_2$, and $e^{(m-\sigma_h)A_2}P_1 \subseteq P_2$.

The relevant norms compute to $\|(A_1 - \sigma_h I)^2\|_{P_1} = 0.0083...$, and $\|(A_2 - \sigma_h I)^2\|_{P_2} = 2.8361...$. Summing up we obtain the bounds $0.0325 < \sigma(\mathcal{A}) < 0.0469$.

In Figure 3 the polytopes P_1 (blue-thick-dashed line) and P_2 (red-thick-dot-dashed line) are plotted, as well the images under the operators $e^{(m-\sigma_h)A_1}$, ...(thin-blue line and thin-red-dot-dashed line).

Example 17. *Let*

$$A_1 = \begin{bmatrix} -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix},$$

and be the dwell time $m = 0.5$.

In Table II we report the discretization length (h), the lower bound σ_h of $\sigma(\mathcal{A})$ given in (2) (lb)*, the upper bound $\sigma_h - \frac{1}{m}\ln\left(1 - \frac{\|(\mathcal{A}-\sigma_h I)^2\|}{8}h^2\right)$ of $\sigma(\mathcal{A})$* (ub)*, and a* leading cycle*, which is a periodic switching law of the discretized system that produces the fastest growth of trajectories. The leading cycle is denoted as $(a_1^1; a_1^2; \ldots; a_k^1; a_k^2)$, where $a_\cdot^1$ and $a_\cdot^2$ are the time of actions of the modes A_1 and A_2 respectively. For example, the leading cycle $(1.3; 1.7)$ in the first line of Table II (the case $h = 0.4$) is the mode A_2 acting the time 1.3 followed by the time 1.7 of the mode A_1. From the table we see that the leading cycle seems to stabilize around $(1.3; 1.7)$ as h increases.*

The character "?" denotes that no leading cycle could be found. On can see, that for too small values of h, the leading cycle cannot be determined, and thus, the lower bound becomes meaningless. Nevertheless, small discretization steps may still give good upper bounds.

TABLE II
RESULTS FOR EXAMPLE 17: THE BOUNDS FOR THE LYAPUNOV EXPONENT AND THE LEADING CYCLES DEPENDING ON h.

h	lb	ub	leading cycle
0.400	0.0762	13.2624	(1.30; 1.70)
0.330	0.0698	8.6828	(1.16; 1.82; 1.49; 1.49)
0.300	0.0751	7.1247	(1.40; 1.70)
0.250	0.0742	4.7571	(1.25; 1.75)
0.200	0.0762	3.0066	(1.30; 1.70)
0.125	0.0762	1.1888	(1.25; 1.625)
0.100	0.0000	0.7298	?
0.050	0.0000	0.3659	?
0.040	0.0000	0.3796	?
0.025	0.0000	0.3125	?

APPENDIX

A. Markovian modified Gripenberg algorithm

The Markovian modified Gripenberg algorithm, is an adaption of the modified Gripenberg algorithm [19]. The main difference is that in each iteration, still, all possible products have to be considered, but just admissible cycles (w.r.t. to the graph) are used to compute the intermediate bounds.

Algorithm 18 (Markovian modified Gripenberg Algorithm)**.** *The algorithm searches for a leading cycle Π_{max} of a graph G and linear operators $\mathcal{E}$.*

$\rho_{max} = 0, \quad \Pi_0 = I$
for $k = 1, \dots, K$
  $\mathcal{E}_k = \Pi_{k-1} \times \mathcal{E}$ *// all possible products in iteration k*
  $\rho_k = \max\{\rho(E)^{1/T(E)} : E \in \mathcal{E}_k \text{ and } E \text{ is a cycle}\}$
    // where $T(E)$ is the time of action of E
  $\Pi_k = \{E \in \mathcal{E}_k : \rho(E)^{1/T(E)} = \rho_k\}$
  if $\rho_k > \rho_{max}$ **then**
    $\Pi_{max} = \Pi_k, \quad \rho_{max} = \rho_k$
  else
    $\Pi_{max} = \Pi_k \cup \Pi_{max}$
  $\Pi_k = \Pi_k^+ \cup \Pi_k^- = \{E \in \mathcal{E}_k : ||E||^{1/T(E)} \geq \rho_k^+\} \cup \{E \in \mathcal{E}_k : ||M||^{1/T(M)} \leq \rho_k^-\}$
    // where $\rho_k^\pm \geq \rho_{max}$ and $\rho_k^\pm$ are such that $\left|\Pi_k^\pm\right| = N/2$
return Π_{max}, ρ_{max}

Example 19. *Given pairs of matrices of various dimensions, random dwell time $m \in (0, 1)$ and random discretization step $h \in (0, m)$. Table III presents the performance of the Markovian modified Gripenberg algorithm* (mg) *compared to a brute force method* (bf)*. For each dimension d we conducted 15 tests and we report in how many cases the Gripenberg like algorithm, or the brute force algorithm worked better Furthermore we give the average runtime of the algorithm (Note though that the Gripenberg like algorithm in general found the final result after approximately $2s$).*

As test matrices we used (a) matrices with random normally distributed entries, and (b) Metzler matrices with random integer entries in $[-9, 9]$, always normalized such that the 2-norm equals 1.

TABLE III
RESULTS FOR EXAMPLE 19: ASSESSMENT OF MARKOVIAN MODIFIED GRIPENBERG ALGORITHM

(a) Random Gaussian matrices

dim	mg better	bf better	t_{mg}	t_{bf}
2	33%	7%	$2.5s$	$13.6s$
3	20%	0%	$2.5s$	$12.7s$
4	33%	7%	$2.4s$	$16.4s$
5	13%	13%	$2.6s$	$16.1s$
7	13%	0%	$2.6s$	$12.9s$
8	20%	0%	$2.6s$	$15.7s$
11	20%	0%	$2.6s$	$13.3s$
13	0%	0%	$2.6s$	$15.9s$
17	0%	0%	$2.6s$	$14.0s$
21	13%	0%	$2.6s$	$17.2s$

(b) Random Metzler matrices

dim	mg better	bf better	t_{mg}	t_{bf}
2	0%	0%	$4.2s$	$27.3s$
3	0%	0%	$2.6s$	$27.5s$
4	40%	0%	$2.6s$	$18.6s$
5	0%	0%	$2.6s$	$26.9s$
7	20%	0%	$2.6s$	$17.6s$
8	40%	0%	$2.5s$	$17.2s$
11	20%	0%	$2.6s$	$13.5s$
13	0%	0%	$2.6s$	$13.4s$
17	40%	0%	$2.5s$	$13.7s$
21	20%	0%	$2.6s$	$14.2s$

One can see that the Gripenberg like algorithm performs in general faster and better than a brute force method.

ACKNOWLEDGMENT

REFERENCES

REFERENCES

[1] N. Barabanov, *Lyapunov indicators of discrete inclusions i–iii*, Autom. Remote Control, 49 (1988), 152–157.

[2] C. Basso, *Switch-mode power supplies spice simulations and practical designs*, McGraw-Hill, Inc., New York, NY, USA, 1 edition, 2008.

[3] F. Blanchini and S. Miani, *A new class of universal Lyapunov functions for the control of uncertain linear systems*, IEEE Trans. Automat. Control, 44 (1999) 3, 641–647.

[4] F. Blanchini, D. Casagrande and S. Miani, *Modal and transition dwell time computation in switching systems: a set-theoretic approach*, Automatica J. IFAC, 46 (2010) 9, 1477–1482.

[5] C. Briat and A. Seuret, *Affine characterizations of minimal and mode-dependent dwell times for uncertain linear switched systems*, IEEE Trans. Automat. Control 58 (2013) 5, 1304–1310.

[6] X. Dai, *Robust periodic stability implies uniform exponential stability of Markovian jump linear systems and random linear ordinary differential equations*, J. Franklin Inst., 351 (2014), 2910–2937.

[7] G. Chesi and P. Colaneri. *Homogeneous rational Lyapunov functions for performance analysis of switched systems with arbitrary switching and dwell time constraints*, IEEE Trans. Automat. Control, 62 (2017) 10, 5124–5137.

[8] G. Chesi, P. Colaneri, J. C. Geromel, R. Middleton, and R. Shorten. *A nonconservative LMI condition for stability of switched systems with guaranteed dwell time*, IEEE Trans. Automat. Control, 57 (2012) 5, 1297–1302.

[9] Y. Chitour, N. Guglielmi, V. Yu. Protasov, and M. Sigalotti, *Switching systems with dwell time: computing the maximal Lyapunov exponent*, Nonlinear Anal. Hybrid Syst. 40 (2021), Paper No. 101021.

[10] A. Cicone, N. Guglielmi, and V. Y. Protasov, *Linear switched dynamical systems on graphs*, Nonlinear Anal. Hybrid Syst., 29 (2018), 165–186.

[11] J.C. Geromel and P. Colaneri, *Stability and stabilization of continuous-time switched linear systems*, SIAM J. Control Optim. 45 (2006) 5, 1915–1930.

[12] N. Guglielmi, L. Laglia, and V. Yu. Protasov, *Polytope Lyapunov functions for stable and for stabilizable LSS*. Found. Comput. Math., 17 (2017) 2, 567–623.

[13] N. Guglielmi and V. Yu. Protasov, *Exact computation of joint spectral characteristics of linear operators*, Found. Comput. Math., 13 (2013), 37–97.

[14] V. Yu. Protasov, and R. Kamalov, *Stability of Continuous Time Linear Systems with Bounded Switching Intervals*, SIAM J. Control Optimiz., 61 (2023) 5, 3051–3075.

[15] V. Kozyakin, *The Berger-Wang formula for the Markovian joint spectral radius*, Linear Alg. Appl., 448 (2014), 315–328.

[16] T. Kröger, *On-line trajectory generation in robotic systems: basic concepts for instantaneous reactions to unforeseen (sensor) events*, Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.

[17] D. Liberzon, *Switching in systems and control*, Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 2003.

[18] D. Liberzon and A. S. Morse, *Basic problems in stability and design of switched systems*, IEEE Control Systems Magazine, 19 (1999), 59–70.

[19] T. Mejstrik, *Improved invariant polytope algorithm and applications*, ACM Trans. Math. Softw., ACM Transactions on Mathematical Software 46 (2020) 3, 1–26.

[20] T. Mejstrik and V. Yu. Protasov, *Elliptic polytopes and invariant norms of linear operators*, Calcolo 60 (2023) 56.

[21] A. Molchanov and Y. Pyatnitskiy, *Criteria of asymptotic stability of differential and difference inclusions encountered in control theory*, Syst. Control Lett., 13 (1989), 59–64.

[22] P. Pepe, *Converse Lyapunov theorems for discrete-time switching systems with given switches digraphs,* IEEE Trans. Autom. Control, 64 (2019) 6, 2502–2508.

[23] A. Pietrus and V. Veliov, *On the discretization of switched linear systems*, Syst. Cont. Letters, 58 (2009) 6, 395–399.

[24] M. Philippe, R. Essick, R. Dullerud, and R. M. Jungers, *Stability of discrete-time switching systems with constrained switching sequences*, Automatica, 72 (2016), 242–250.

[25] M. Philippe and R. M. Jungers, *A sufficient condition for the boundedness of matrix products accepted by an automaton*, Proceedings of the 18th International Conference on Hybrid Systems: Computation and Control, ACM New York, NY, USA (2015), 51–57.

[26] M. Philippe, G. Millerioux, and R. M. Jungers, *Deciding the boundedness and dead-beat stability of constrained switching systems*, Nonlinear Anal. Hybr. Syst., 23 (2017), 287–299.

[27] V. Yu.Protasov, *Generalized Markov-Bernstein inequalities and stability of dynamical systems*, Proc. Steklov Inst., 319 (2022), 237–252.

[28] V. Yu. Ptotasov and R. M. Jungers, *Analysing the stability of linear systems via exponential Chebyshev polynomials,* IEEE Trans. Autom. Control 61 (2016) 3, 795–798.

[29] F. Rossi, P. Colaneri, and R. Shorten, *Padé discretization for linear systems with polyhedral lyapunov functions*, IEEE Trans. Autom. Control, 56 (2011) 11, 2717–2722.

[30] R. Shorten, M. Corless, S. Sajja and S. Solmaz, *On Padé approximations, quadratic stability and discretization of switched linear systems*, Syst. & Contr. Letters, 60 (2011) 9, 683–689.

[31] M. Souza, A. Fioravanti, and R. Shorten, *Dwell-time control of continuous-time switched linear systems*, In Proceedings of the 54th IEEE Conference on Decision and Control (2015), 4661–4666.

[32] A. van der Schaft and H. Schumacher, *An introduction to hybrid dynamical systems*, vol. 251 of Lecture Notes in Control and Information Sciences, Springer-Verlag London, Ltd., London, 2000.

[33] W. Xiang, *On equivalence of two stability criteria for continuous-time switched systems with dwell time constraint*, Automatica J. IFAC, 54 (2015), 36–40.

Thomas Mejstrik received his Master in piano studies from University of music and performing arts Vienna and earned his Doctor of Sciences in Mathematics for work on joint spectral radius and subdivision schemes from the University of Vienna, Austria, in 2019. Since 2019 he is researcher at Dimetor GmbH and since 2020 a post doc at the University of Vienna, Austria, wireless networks and music.